Prediction Intervals

Keegan Korthauer
Department of Statistics
UW Madison

What is a Prediction Interval?

- All of Chapter 5 so far has dealt with confidence intervals
  - recall that a CI is an interval likely to contain the true value of the **population parameter** (e.g. μ, p)
  - we learned how to construct CIs in many situations: large vs small sample, one vs two samples, mean vs proportion, and paired data
- Suppose that we want an interval that is likely to contain the **value of a future sample** (instead of a population parameter)
  - then we need a **prediction interval**

In Other Words…

Confidence Interval = interval likely to contain population parameter

Prediction Interval = interval likely to contain the next sampled data point

Deriving the PI

- Say we already sampled a small set of n objects ($n < 30$)
- Assume that the population is known to be normal with population mean μ and population variance σ^2
- Then the sample mean is normal: $\overline{X} \sim N(\mu, \sigma^2/n)$
- Let Y be a single future observation: then $Y \sim N(\mu, \sigma^2)$
- Then $Y - \overline{X}$ is a linear combination of two normal RVs, so

$$Y - \overline{X} \sim N(0, \sigma^2(1+1/n))$$

Deriving the PI - Continued

- But, σ^2 is unknown, and we have a small sample, so we must use the t distribution

$$\frac{Y-\bar{X}}{s\sqrt{1+\frac{1}{n}}} \sim t_{n-1} \Rightarrow P\left(-t_{n-1,\ \alpha/2} < \frac{Y-\bar{X}}{s\sqrt{1+1/n}} < t_{n-1,\ \alpha/2}\right) = 1-\alpha$$

- Our best point estimate for Y is $\bar{X}$

- There is uncertainty in both Y and $\bar{X}$

Summary: Prediction Intervals

Let $X_1,\ldots,X_n$ be a sample from a **normal** population. Let Y be an additional item to be sampled from this population (not yet observed).

Then a level 100(1-α)% **prediction interval** for Y is

$$\bar{X} \pm t_{n-1,\ \alpha/2}\, s\sqrt{1+\frac{1}{n}}$$

Notes about PIs

- Only valid when the population is normal
- What can we say about the width of a prediction interval compared to the width of a confidence interval?
- One-sided prediction intervals (upper/lower prediction bounds) can be obtained by replacing $\alpha/2$ with α and $\pm$ with + (upper) or – (lower)

Example – Alloy Adhesion

An article in the journal *Materials Engineering* (1989) presents the results of tensile adhesion tests on 22 U-700 alloy specimens

The load at specimen failure is as follows (in megapascals):

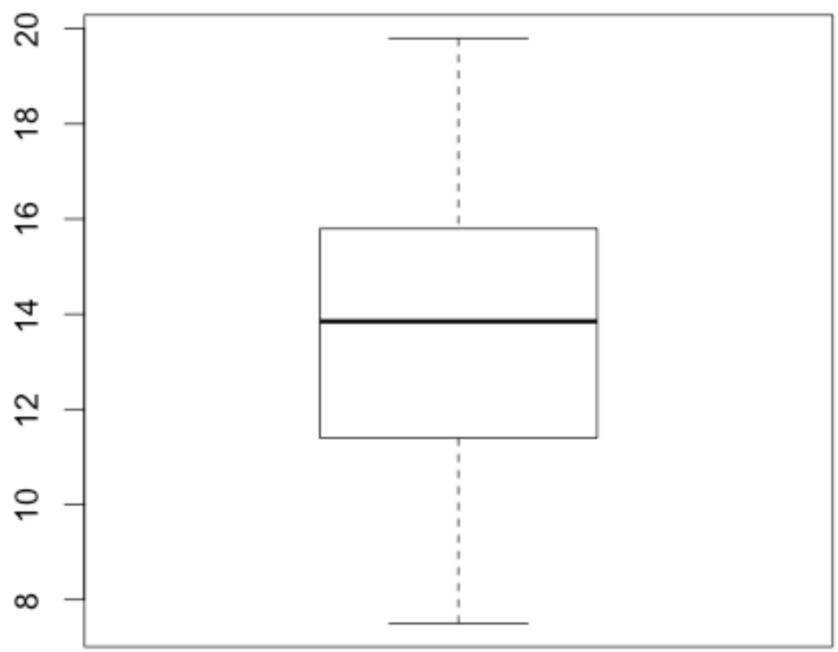
19.8 10.1 14.9 7.5 15.4 15.4 15.4 18.5 7.9 12.7 11.9
11.4 11.4 14.1 17.6 16.7 15.8 19.5 8.8 13.6 11.9 11.4

Sample mean is 13.71, sample standard deviation is 3.55

We plan to test a 23^{rd} specimen and want a 95% PI on the load at failure for this specimen.

Example – Alloy Adhesion

First step – is my sample coming from a normal distribution?



Example – Alloy Adhesion

Next – Construct the PI: Let Y represent the load at failure for the new specimen.

Then a 95% PI for Y is:

$$\bar{X} \pm t_{21,\,0.025}\, s\sqrt{1+\frac{1}{22}}$$

How does this compare with a 95% CI for the population mean?

$$= 13.71 \pm 2.080 * 3.55\sqrt{1+\frac{1}{22}}$$

$$= 13.71 \pm 7.55 = [6.16,\ 21.26]$$

Next

- Note that we are not covering ‘Tolerance Intervals’ (also included in section 5.8)

- Chapter 6 – Hypothesis Testing

- Homework 6 due on Friday