More Continuous Distributions

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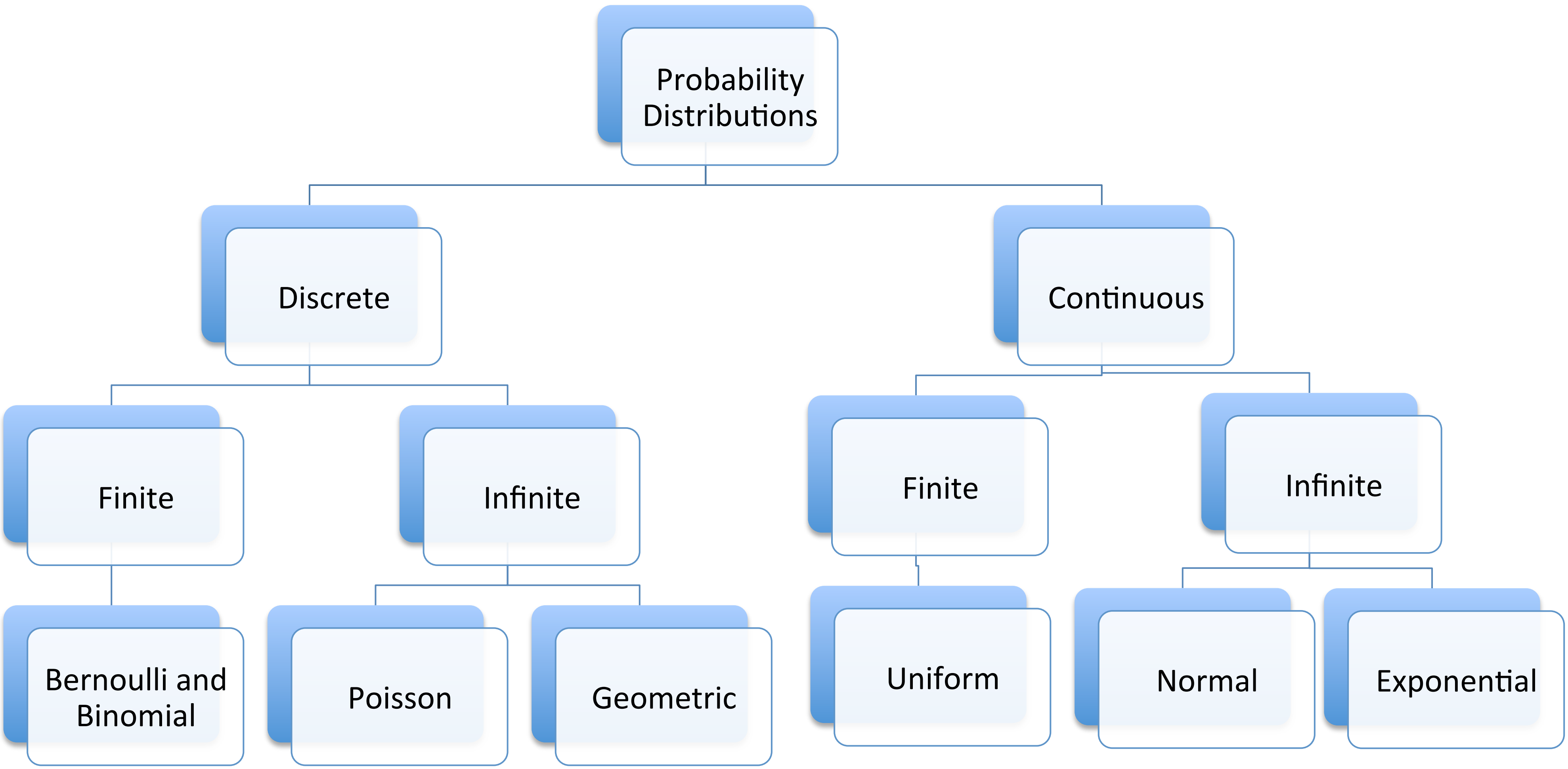
Some Common Distributions



CONTINUOUS DISTRIBUTIONS

Normal distribution

Continuous Uniform Distribution

Exponential distribution

Uniform Distribution

- A normally distributed RV tends to be close to its mean
- Instead consider a random variable X that is equally likely to take on *any* value in a continuous interval (*a,b*)
  - Example: waiting time for a train if we know it will arrive within a certain time range but have no prior knowledge beyond that
- Then X is distributed **uniformly** on the interval (*a,b*)
- The Uniform distribution is especially useful in computer simulations

X ~ Uniform(a,b)

- Probability Density Function

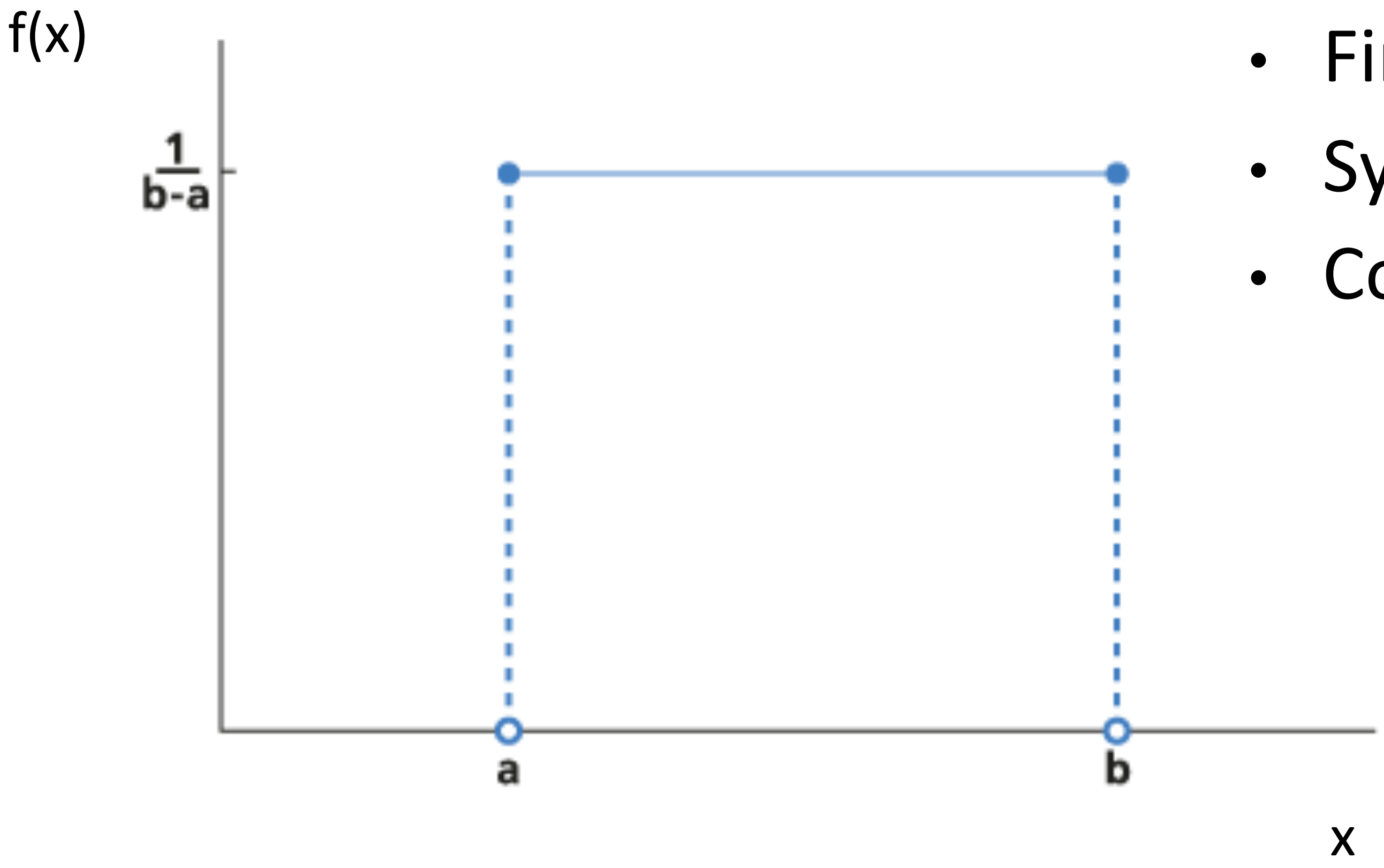
$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

- Mean and Variance

$$\mu_X = \frac{a+b}{2}$$

$$\sigma_X^2 = \frac{(b-a)^2}{12}$$

Uniform PDF



- Continuous
- Finite
- Symmetric
- Constant

Example – Waiting Time

- Suppose we know that the time a train will arrive is distributed uniformly between 5:00pm and 5:15pm

- Let X represent the number of minutes we have to wait if we arrive at 5:00pm.

- What is the expected time we will wait?

    7.5 minutes

- What is the probability that we will wait longer than 10 minutes?

    1/3

Infinite Waiting Times?

- Say we want to model the lifetime X of a component and we do not have an upper bound
- We want to allow for the possibility that the component will last a really long time (let X approach infinity)
- Our PDF still has to integrate to one, so it will have to asymptotically decrease to zero as X goes to infinity
- In this case, we can model X as an **exponential** random variable with shape parameter λ
- Can think of the exponential as the continuous analog to the geometric distribution

X ~ Exponential(λ)

- Probability Density Function

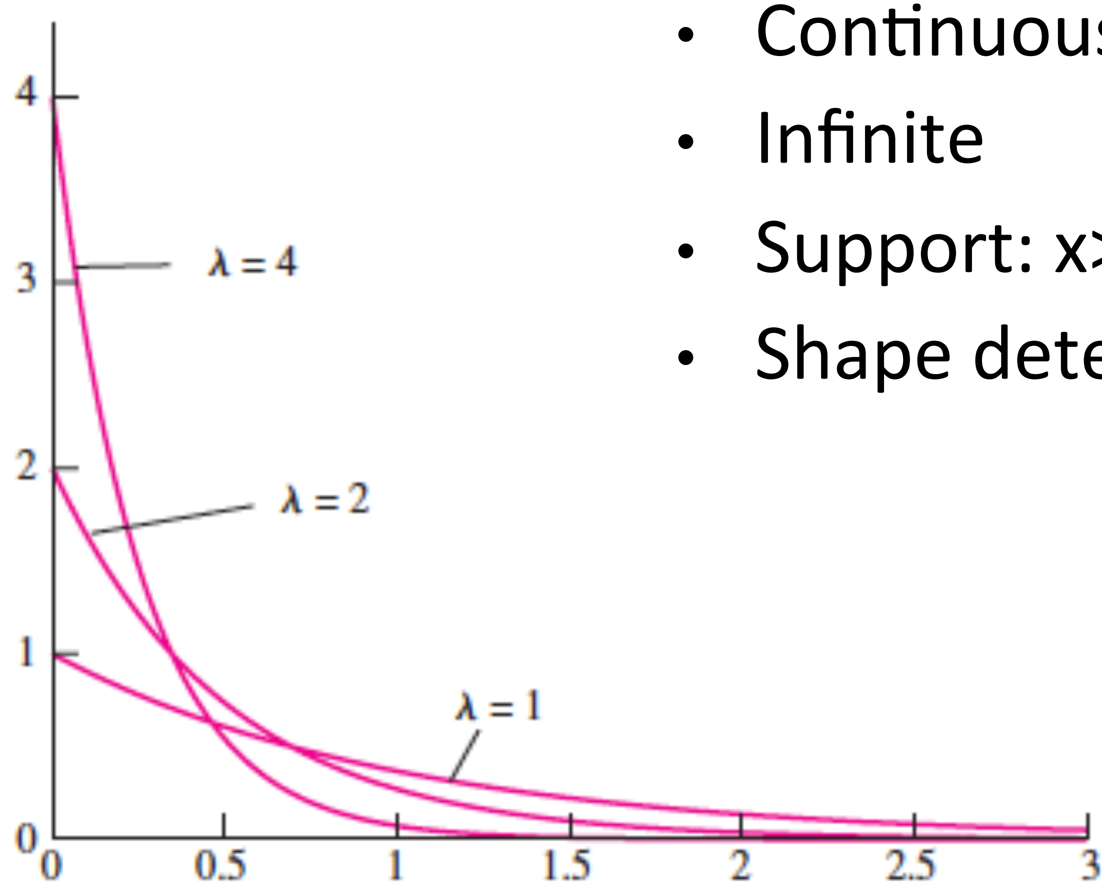
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Mean and Variance

$$\mu_X = \frac{1}{\lambda}$$

$$\sigma_X^2 = \frac{1}{\lambda^2}$$

Exponential Distribution



- Continuous
- Infinite
- Support: x>0
- Shape determined by λ

FIGURE 4.17 Plots of the exponential probability density function for various values of λ.

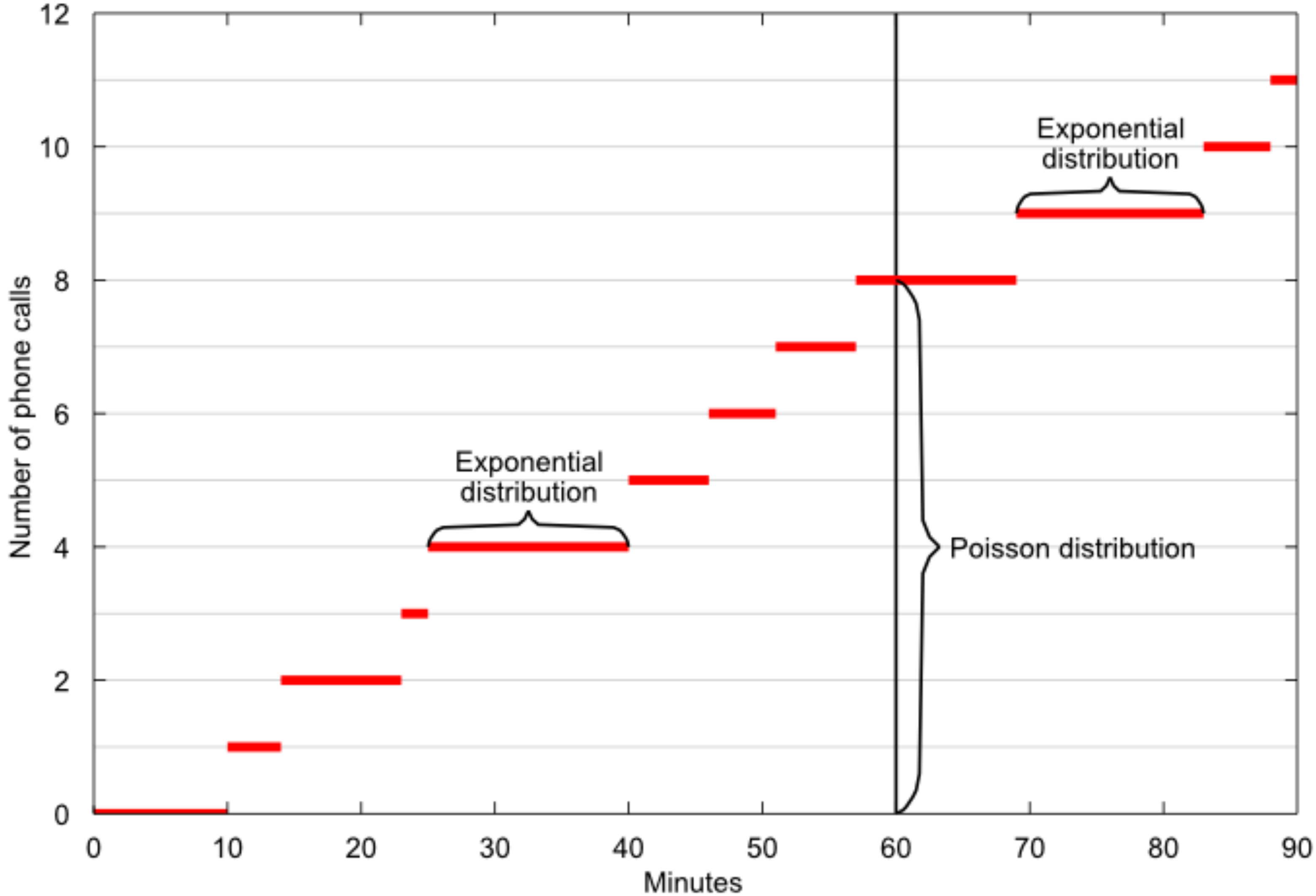
Connection to the Poisson

- Recall that the Poisson distribution models the number of events occurring within one unit of time or space given some rate parameter λ

- If instead of counting the *number of events* that occur we are interested in modeling the *time between two events*, we can use the exponential distribution

- Formally, if events follow a Poisson process with rate parameter λ, then the waiting time T from any starting point until the next event time is distributed Exp(λ)

Calling Center Example



http://www.statlect.com/uddpoi1.htm

Example 4.58

- A radioactive mass emits particles according to a Poisson process at a rate of 15 particles per minute. At some point, we start a stopwatch.

- What is the probability that more than 5 seconds will elapse before the next emission?

  P(T>5) = 0.2865

- What is the mean waiting time until the next particle is emitted?

  E(T) = 4 seconds

Lack of Memory Property

- The exponential distribution does not 'remember' how long we've been waiting
- The time until the next event in a Poisson process *from any starting point* is exponential
- The distribution of the waiting time is the same whether the time period just started or we have already waited *t* time units
- We call this **memorylessness** or the **lack of memory property**

If $T \sim Exp(\lambda)$ then $P(T > t \mid T > s) = P(T > t)$ for $s, t > 0$

Example 4.59-60 – Circuit Lifetime

- The lifetime of a particular circuit has an exponential distribution with mean 2 years.

- Find the probability that the circuit lasts longer than 3 years.

- Assume the circuit is already 4 years old. What is the probability that it will last more than 3 additional years?

When λ is Unknown

How can we estimate it?

- If X ~ Exp(λ) then $\mu_X = 1/\lambda$ so $\lambda = 1/\mu_X$
- Then we can estimate λ with $\hat{\lambda} = \dfrac{1}{\bar{X}}$
- Problem - although μ_X is unbiased for the sample mean, $1/\mu_X$ is **not** a linear function of μ_X so our estimator is biased:

$$E(\hat{\lambda}) \approx \lambda + \lambda / n$$

- For large samples, the bias will be negligible but for small samples we can correct for the bias with the estimator

$$\hat{\lambda}_{corr} = \frac{1}{\bar{X} + \bar{X}/n} = \frac{n}{(n+1)\bar{X}}$$

When λ is Unknown

What is the uncertainty in our estimate?

- Using the propagation of error method from Chapter 3, we can estimate it with

$$\sigma_{\hat{\lambda}} \approx \left| \frac{d}{d\bar{X}} \frac{1}{\bar{X}} \right| \sigma_{\bar{X}} = \frac{1}{\bar{X}^2} \frac{1}{\lambda\sqrt{n}} \overset{\text{plug in } \lambda=\hat{\lambda}}{=} \frac{1}{\bar{X}\sqrt{n}}$$

- Because of the bias in the estimate of λ, we will underestimate the uncertainty when we have a small sample size

Next

- Skim section 4.6 (the lognormal distribution) and the rest of section 4.8 (the gamma and Weibull distributions)

- Exam 1 (next Friday 2/28) covers material up to and including this lecture
  - Practice exam handed out this Friday
  - Review session on Wednesday

- In the next few lectures we will
  - finish up Chapter 4 -Probability Plots and the Central Limit Theorem
  - introduce the statistical language R