Propagation of Error

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Error Propagation

- Error is inherent in any measuring procedure
- Measured values will differ from the underlying (often unknown) true value
- Calculations performed on these measurements will also contain error (i.e. they are **propagated** from the measurement to the calculated value)

Error in Measurements
e.g. length and width of a rectangle

→ **Calculation using Measurements**
e.g. length x width

→ **Error in Calculated Value**
e.g. area of a rectangle

- There are ways to estimate the likely size of the error in a calculation from the size of the measurement error

MEASUREMENT ERROR

Bias vs Random Error

Estimating Uncertainty

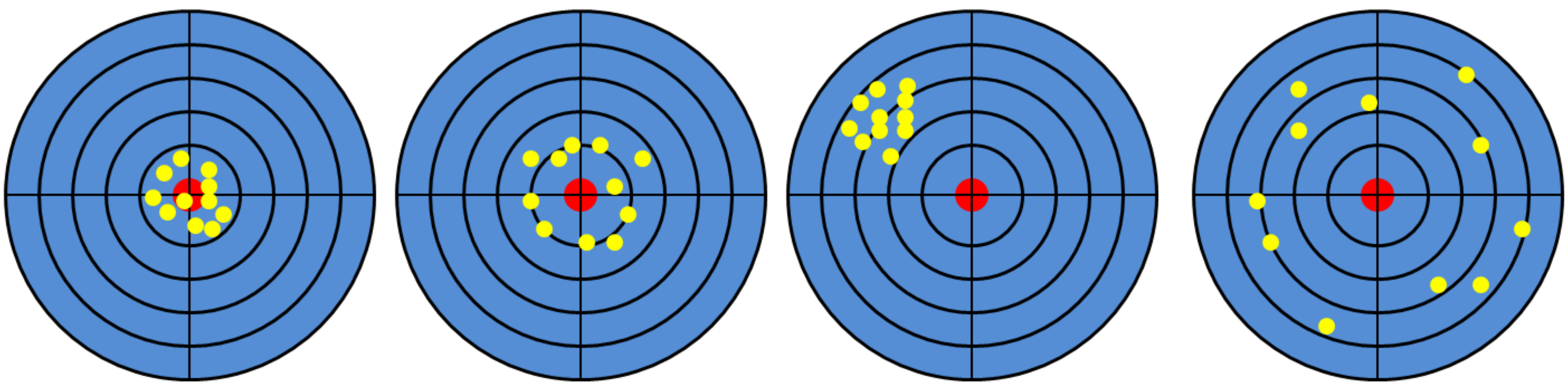
Bias and Random Error

$$\text{Measured Value} = \text{True Value} + \underbrace{\boxed{\text{Bias} + \text{Random Error}}}_{\text{Total Error}}$$

- Bias (also known as Systematic Error)
  - Same for every measurement
  - Example: lab tech forgets to tare scale when weighing product in a dish
- Random Error
  - Varies from measurement to measurement
  - Likely to average out to zero in the long run
  - Example: Rounding or interpolating between graduation marks

Accuracy and Precision

- **Accuracy** is how close the mean value μ is to the true value being measured (determined by bias)
- **Precision** is the degree to which repeated measurements tend to agree with each other (determined by variance)



Accurate & Precise

Accurate & Imprecise

Inaccurate & Precise

Inaccurate & Imprecise

www.knowitsdifference.com

Summary

For a measured random variable with mean μ and standard deviation σ:

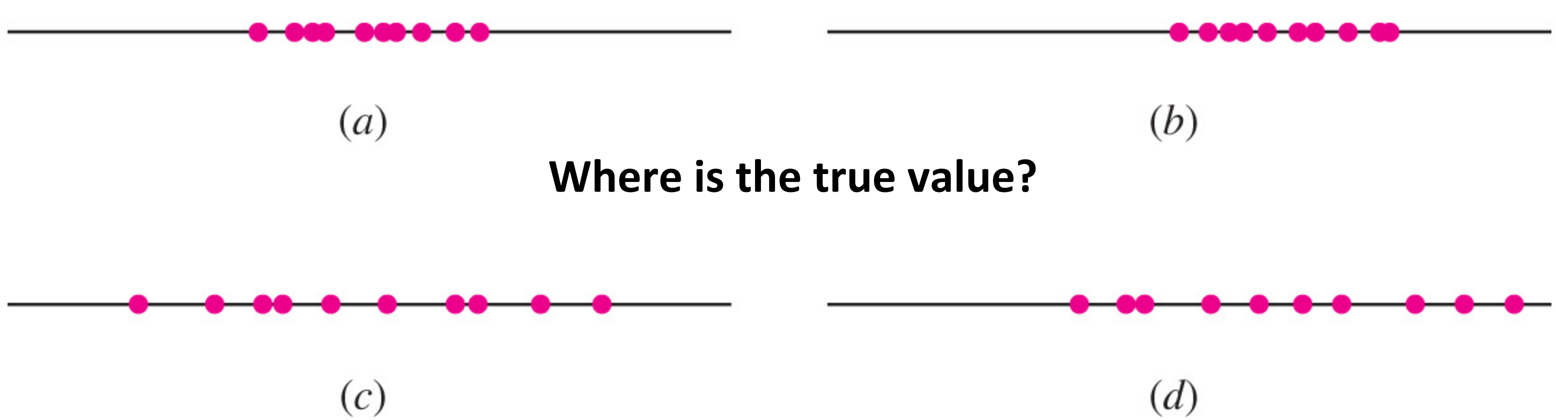
- **Bias** = μ – true value
- **Uncertainty** (size of typical random error) = σ
- **Accuracy** corresponds to low bias
- **Precision** corresponds to low uncertainty

Estimation of Bias

- An estimate of bias is the difference between the true value and the average of *repeated measurements of the same quantity*

$$\text{Bias} \approx \text{Sample Mean} - \text{True Value} = \bar{X} - \text{True Value}$$

- When we do not know the true value that we are measuring (usually the case), we cannot estimate bias



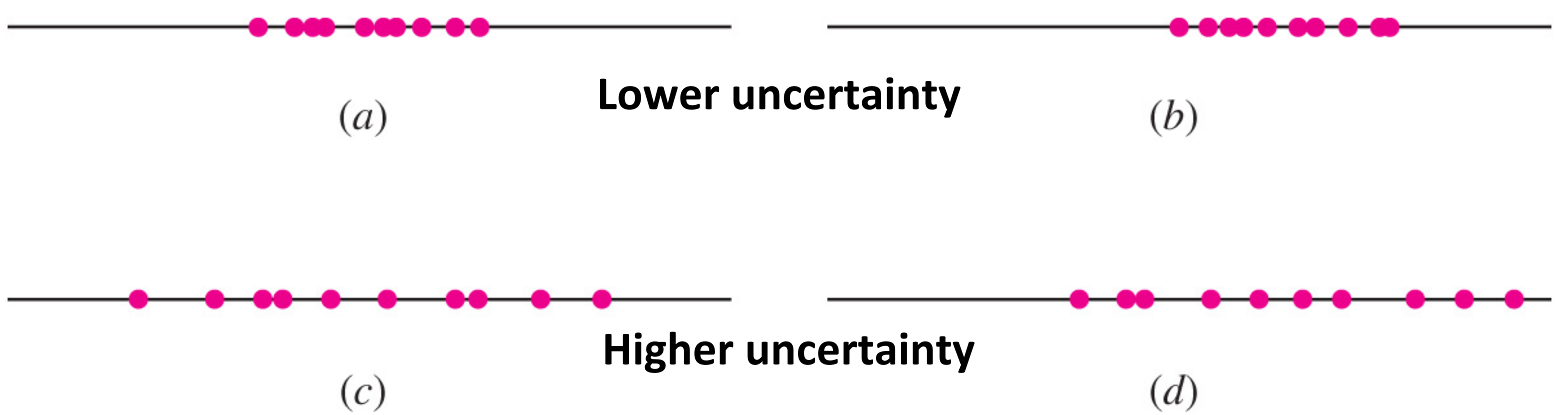
Where is the true value?

Estimation of Uncertainty

- We can estimate uncertainty as the standard deviation of *repeated measurements of the same quantity*

Uncertainty ≈ Sample Standard Deviation = σ

- It is best to have a large sample size for a good estimate of σ



Example – Exercise 3.1.7

- A replica of The Kilogram (the standard weight that represents exactly 1kg) is weighed five times
- The measurements are (in units of micrograms above 1kg):

114.3, 82.6, 136.4, 126.8, 100.7

(a) Can we estimate the uncertainty?

(b) Can we estimate the bias?

Example – Exercise 3.1.7

- A replica of The Kilogram (the standard weight that represents exactly 1kg) is weighed five times
- The measurements are (in units of micrograms above 1kg):

114.3, 82.6, 136.4, 126.8, 100.7

(a) Can we estimate the uncertainty?

Yes, estimate with sample sd = 21.3 micrograms

(a) Can we estimate the bias?

No, since we do not know the true value

UNCERTAINTY OF A LINEAR COMBINATION OF MEASUREMENTS

Linear Combinations of Independent Repeated Measurements

Linear Combinations of Dependent Repeated Measurements

This should look familiar…

- How do we estimate uncertainty for a measurement multiplied by a constant? (e.g. unit conversion)
- How do we estimate uncertainty for the sum of two or more measurements? (e.g. adding commute times)
- Measurements are **random variables** so we can apply what we learned in Chapter 2, section 5 about finding variances of linear combinations of random variables

Summary of Uncertainty for Linear Combinations of Measurements

- Let $X_1,\dots,X_n$ be **independent** measurements and $c_1,\dots,c_n$ be a constants. Then,

$$\mathrm{sd}(c_1X_1+\dots+c_nX_n)=\sigma_{c_1X_1+\dots+c_nX_n}=\sqrt{c_1^2\sigma_{X_1}^2+\dots+c_n^2\sigma_{X_n}^2}$$

- If $X_1,\dots,X_n$ are **dependent**, then we can only get an upper bound on the uncertainty of the linear combination

$$\mathrm{sd}(c_1X_1+\dots+c_nX_n)=\sigma_{c_1X_1+\dots+c_nX_n}\le|c_1|\sigma_{X_1}+\dots+|c_n|\sigma_{X_n}$$

Example 3.6

Suppose a surveyor needs to know the perimeter of a rectangular lot and takes repeated measures of two adjacent sides and finds that side X has a mean of 50.11m and an sd of 0.05m. Side Y has a mean of 75.21m with an sd of 0.08m. Assume measurements X and Y are independent.

What is the uncertainty in the estimate of the perimeter?

Example 3.6

Suppose a surveyor needs to know the perimeter of a rectangular lot and takes repeated measures of two adjacent sides and finds that side X has a mean of 50.11m and an sd of 0.05m. Side Y has a mean of 75.21m with an sd of 0.08m. Assume measurements X and Y are independent.

Perimeter Estimate: $P = 2X + 2Y = 250.64\text{m}$

The uncertainty in P is:

$$\sigma_P = \sigma_{2X+2Y} = \text{sqrt}(\sigma^2_{2X+2Y})$$

$$= \text{sqrt}(2^2\sigma^2_X + 2^2\sigma^2_Y)$$ <- (Since X and Y are independent)

$$= \text{sqrt}(4(0.05)^2 + 4(0.08)^2)$$

$$= 0.19\text{m}$$

Example 3.6 Continued

The surveyor's assistant came up with a different result. His solution:

Perimeter Estimate: P = 2X + 2Y = = X + X + Y + Y, so the uncertainty in P is:

$$\sigma_P = \sigma_{X+X+Y+Y} = \text{sqrt}(\sigma^2_{X+X+Y+Y})$$
$$= \text{sqrt}(\sigma^2_X + \sigma^2_X + \sigma^2_Y + \sigma^2_Y)$$
$$= \text{sqrt}((0.05)^2 + (0.05)^2 + (0.08)^2 + (0.08)^2)$$
$$= 0.13\text{m}$$

What happened here?

Example 3.6 Continued

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Perimeter Estimate: $P = 2X + 2Y = = X + X + Y + Y$, so the uncertainty in P is:

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Assumes independence!

$$= \text{sqrt}(\sigma^2_X + \sigma^2_X + \sigma^2_Y + \sigma^2_Y)$$

$$= \text{sqrt}((0.05)^2 + (0.05)^2 + (0.08)^2 + (0.08)^2)$$

$$= 0.13\text{m}$$

What happened here?

Commute Time Example Revisited

- Let X_1 represent the time it takes (in minutes) to walk from my house to the bus stop. Assume $E(X_1)=3$, $Var(X_1)=1$.
- Let X_2 represent the time it takes the bus to travel between the bus stop and campus. Assume $E(X_2)=8$, $Var(X_2)=4$.
- What can we say about the uncertainty of $Y= X_1 + X_2$ if the two legs are NOT independent?

$$\sigma_{X_1+X_2} \leq |1|\sigma_{X_1} + |1|\sigma_{X_2}$$

$$= \sqrt{\sigma_{X_1}^2} + \sqrt{\sigma_{X_2}^2}$$

$$= \sqrt{4} + \sqrt{1} = 3$$

UNCERTAINTIES OF FUNCTIONS OF MEASUREMENTS

Functions of one measurement

Functions of several independent measurements

Functions of several dependent measurements

Uncertainty for a (Nonlinear) Function of a Measurement

- Recall that given a RV X and an arbitrary function U=U(X) we can find the mean of U(X) if we have its PMF (if discrete) or PDF (if continuous):

$$E[U(X)] = \sum_x U(x)p(x) \quad \text{or} \quad E[U(X)] = \int_{-\infty}^{\infty} U(x)f(x)\,dx$$

- How do we estimate the uncertainty of U(X), σ_U?
- If X is a measurement with uncertainty σ_X and σ_X is small, and if X is estimated by x, then

$$\sigma_U \approx \left| \left(\left. \frac{dU}{dX} \right|_{X=x} \right) \right| \sigma_X$$

Uncertainty for a (Nonlinear) Function of a Measurement

This equation is known as the **propagation of error** formula

$$\sigma_U \approx \left| \left(\frac{dU}{dX}\bigg|_{X=x} \right) \right| \sigma_X$$

Notes:

(a) it is only an approximation (see pgs 183-184 for derivation) [Taylor series approx of U ≈ constant + dU/dX*(X)]

(a) the estimate can be biased if either (1) σ_X is large or (2) the second derivative of U is large or (3) the estimate x is biased

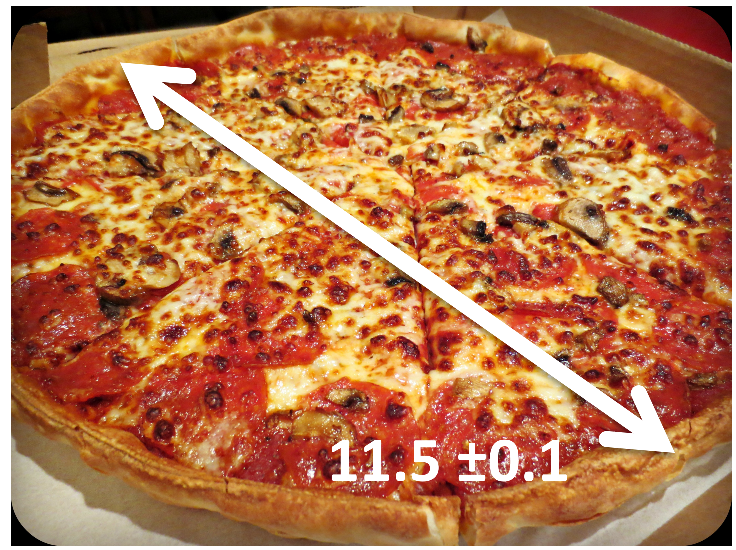
(b) we will operate under the assumption that the bias is negligible for the purposes of this course

Example – Area of a Pizza

We've measured the diameter D of a pizza several times and discovered that the average was 11.5 inches and the sd was 0.1 inches.

Estimate the area of the pizza and

find the uncertainty in that estimate.



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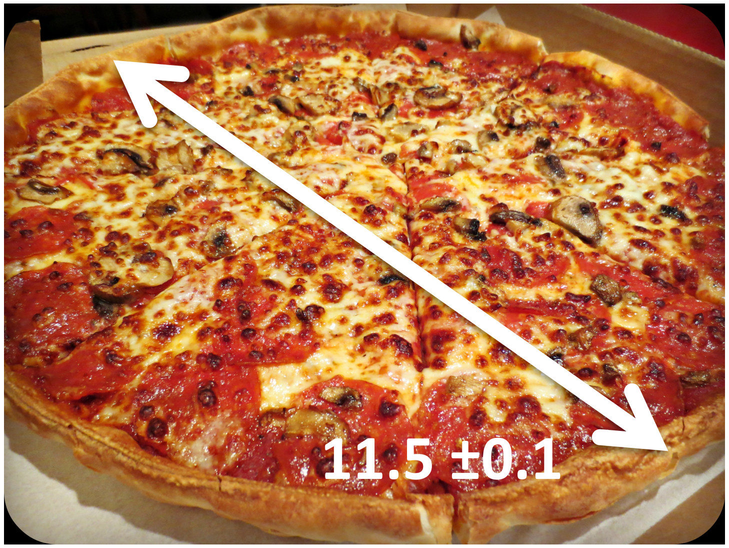

Estimate: A = 103.87in^2

Uncertainty: $\sigma_A \approx 1.81$ in^2

Uncertainties for (Nonlinear) Functions of Several Measurements

- The propagation of error formula generalizes for multiple **independent** measurements $X_1,\ldots,X_n$ estimated by $x_1,\ldots,x_n$ with (small) uncertainties $\sigma_{X1},\ldots,\sigma_{Xn}$ and a function $U = U(X_1,\ldots,X_n)$:

$$\sigma_U \approx \sqrt{\left(\left.\frac{\partial U}{\partial X_1}\right|_{\vec{X}=\vec{x}}\right)^2 \sigma_{X_1}^2 + \ldots + \left(\left.\frac{\partial U}{\partial X_n}\right|_{\vec{X}=\vec{x}}\right)^2 \sigma_{X_n}^2}$$

$$\text{where } \vec{X} = (X_1,\ldots,X_n) \text{ and } \vec{x} = (x_1,\ldots,x_n)$$

- If the measurements are NOT independent, we can only get an estimate of the upper bound for the uncertainty

$$\sigma_U \leq \left|\left(\left.\frac{\partial U}{\partial X_1}\right|_{\vec{X}=\vec{x}}\right)\right| \sigma_{X_1} + \ldots + \left|\left(\left.\frac{\partial U}{\partial X_n}\right|_{\vec{X}=\vec{x}}\right)\right| \sigma_{X_n}$$

Example 3.18 – Density of a Rock

The mass (m) and volume (V) of a rock are measured repeatedly and found to be ***m* = 674.0 ± 1.0g and V = 261.0 ± 0.1mL**. Assume that *m* and V are independent.

Estimate the density D and find its uncertainty.

What if *m* and V are not independent?

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Estimate the density D and find its uncertainty.

2.582 ± 0.004 g/mL

What if *m* and V are not independent?

$\sigma_D \leq 0.0048$ g/mL

Next

- Note: we are not going to cover the topic of relative uncertainties (pg 182 for single measurements and pg 190 for multiple measurements)

- Chapter 4 – Distributions!