Joint Distributions

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JOINT DISTRIBUTIONS OF MORE THAN ONE RANDOM VARIABLE

Jointly Discrete vs Jointly Continuous

Marginal distribution

Dealing with Multiple RVs

- So far we've dealt with linear combinations of multiple random variables
  - addition/multiplication by constants
  - sum/subtract to create a new RV
- New idea: each item in the population has more than one random variable associated with it
- We want to study how these **vary together**
- Example – age and income level in a certain town
  - Do people of certain age levels make different amounts of money?

Jointly Discrete RVs

- Two discrete random variables X and Y associated with each item in a population are called **jointly discrete**
- The **joint probability mass function** (or joint PMF) specifies the probability of each possible value of the ordered pair (X,Y)

$$p(x,y) = P(X = x \text{ and } Y = y)$$

Property of the Joint PMF

The joint probability mass function of jointly discrete RVs X and Y must sum to one when adding up the probabilities of all possible ordered pairs (X,Y):

$$\sum_{x}\sum_{y} p(x,y) = 1$$

Example – Two Discrete RVs

- Let X be the number of cars in a household of a certain community and let Y be the number of children in a household of the same community
- X and Y are jointly discrete
- Let the PMF be:

		Y = Number of Cars	
		1	**2**
X = Number of Children	**0**	0.15	0.05
	1	0.10	0.50
	2	0.05	0.15

- What is the probability that a randomly chosen household has no children and 2 cars?
- What is $P(X \geq 1 \text{ and } Y < 2)$?

Marginal Probability

- Sometimes we have the joint PMF of two discrete RVs X and Y, but we are only interested in one of them
- The **marginal** probability mass functions of X alone and Y alone when we are only given the joint PMF p(x,y) are

$$p_X(x) = P(X = x) = \sum_y p(x,y), \text{ and}$$

$$p_Y(y) = P(Y = y) = \sum_x p(x,y)$$

Example – Marginal PMF

- Find the marginal probability mass function for Y, the number of cars

		Y = Number of Cars	
		1	2
X = Number of Children	0	0.15	0.05
	1	0.10	0.50
	2	0.05	0.15

$$p(Y=1)=\sum_{x=0}^{2} p(x,1)=0.15+0.10+0.05=0.3$$

$$p(Y=2)=\sum_{x=0}^{2} p(x,2)=0.05+0.50+0.15=0.7$$

Jointly Continuous RVs

- Two continuous random variables X and Y associated with each item in a population are called **jointly continuous**
- Probabilities are found by integrating the **joint probability distribution function** (or joint PDF) of two variables f(x,y)
- Then for a<b and c<d,

$$P(a \le X \le b \text{ and c} \le Y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Property of the Joint PDF

The joint probability distribution function of jointly continuous RVs X and Y must integrate to one over the entire sample space of X and Y:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

Example 2.54

For a certain type of washer, the thickness and diameter varies from item to item. Let X be the thickness (in mm) and let Y be the diameter (in mm). Assume the joint PDF of X and Y is:

$$f(x,y)=\begin{cases}\frac{1}{6}(x+y) & 1\le x\le 2 \text{ and } 4\le y\le 5\\ 0 & \text{otherwise}\end{cases}$$

Find the probability that a randomly chosen washer has a thickness between 1.0 and 1.5 mm and a diameter between 4.5 and 5.0 mm

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Find the probability that a randomly chosen washer has a thickness between 1.0 and 1.5 mm and a diameter between 4.5 and 5.0 mm = 0.25

Marginal Probability

- Sometimes we are given the joint PDF of two RVs but we are only interested in one of them
- The **marginal** probability distribution functions of X alone and Y alone when we are only given the joint PDF f(x,y) are

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \text{ and}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Example 2.54 Continued

Given the joint PDF of X and Y:

$$f(x,y) = \begin{cases} \frac{1}{6}(x+y) & 1 \le x \le 2 \text{ and } 4 \le y \le 5 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal probability density functions of X and Y

Example 2.54 Continued

Given the joint PDF of X and Y:

$$f(x,y)=\begin{cases}\frac{1}{6}(x+y) & 1\le x\le 2 \text{ and } 4\le y\le 5\\ 0 & \text{otherwise}\end{cases}$$

Find the marginal probability density functions of X and Y

$f_X(x) = (x + 4.5)/6$ if $1 \le x \le 2$

$f_Y(y) = (y + 1.5)/6$ if $4 \le y \le 5$

Next

- Note that we are only covering section 2.6 up to page 137 (no conditional distributions or conditional expectation)

- Measurement Error (3.1)

- Uncertainties for linear combinations of measurements (3.2)