Counting Methods and Conditional Probability

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COUNTING METHODS

Permutations

Combinations

Permutation

- Definition: an ordering of a collection of objects

- Counting – How many ways can we arrange n objects?

n	Objects	Ordered
2	A, B	AB, BA
3	A, B, C	ABC, ACB, BAC, BCA, CAB, CBA
4+	A, B, C, D, …	?

Permutation

- How many ways can we order 4 items (A,B,C,D)?
  - 4 ways to pick the 1st item
  - 3 ways to pick the 2nd item
  - 2 ways to pick the 3rd item
  - 1 way to pick the 4th item
- By the Fundamental Principle of Counting, there are

  __4__ x __3__ x __2__ x __1__ = 24 ways
- This pattern generalizes – the number of permutations of n objects is

  n(n-1)(n-2) … (3)(2)(1) = n! , or "n factorial"

Example – Permutations of Subsets

- There are eight people running in a 400 meter race
- They are competing for three medals (gold, silver, and bronze)
- How many ways can the three medalists be chosen and ordered on the podium?

Example Continued

- By the Fundamental Principle of Counting, the number of ways 3 medalists can be ordered on the podium is

  _8_ x _7_ x _6_ = 336

- In general, the number of permutations of k objects chosen from a group of n objects is

$$= \frac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots(3)(2)(1)}{(n-k)(n-k-1)\cdots(3)(2)(1)}$$

$$= \frac{n!}{(n-k)!}$$

Combination

- Each distinct group of objects that can be selected **without regard to order**
- Previous example – say we are only interested in whether a runner makes the podium or not (place doesn't matter)
  - The following outcomes are all equivalent

    ABC, ACB, BAC, BCA, CAB, CBA
  - Likewise, there will be 6 possible permutations for any group of 3 runners making the podium

Combination

- We already know that the number of permutations of k objects chosen from n is n!/(n-k)!

- Also, the number of permutations of k objects is k!

- Therefore, the number of **combinations** of k objects chosen from n is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

"n choose k"

Example 2.14

- At a certain event, 30 people attend, and 5 will be chosen at random to receive door prizes. The prizes are all the same, so the order in which the people are chosen does not matter.

- How many different groups of 5 people can be chosen?

$$\binom{30}{5} = \frac{30!}{5!25!} = \frac{(30)(29)(28)(27)(26)}{(5)(4)(3)(2)(1)} = 142{,}506$$

Combination – Multiple Subsets

- Choosing a combination of k objects from n divides the n objects into 2 subsets:

- We can also divide the n objects into multiple subsets

Example

Consider the task of dividing the class of 144 students into 6 discussion groups of 24 each	Number of Ways of Performing the Operation
Choose a combination of 24 students	$\binom{144}{24} = \frac{144!}{24!120!}$
Choose 24 students from the remaining 120	$\binom{120}{24} = \frac{120!}{24!96!}$
Choose 24 students from the remaining 96	$\binom{96}{24} = \frac{96!}{24!72!}$
Choose 24 students from the remaining 72	$\binom{72}{24} = \frac{72!}{24!48!}$
Choose 24 students from the remaining 48	$\binom{48}{24} = \frac{48!}{24!24!}$

Example Continued

- By the Fundamental Principle of Counting, multiply these numbers to get the total number of combinations:

$$\binom{144}{24}\binom{120}{24}\binom{96}{24}\binom{72}{24}\binom{48}{24} = \frac{144!}{24!24!24!24!24!} = 9.7 \times 10^{106}$$

- Generalization: The number of ways of dividing a group of n objects into groups of $k_1,\ldots,k_r$ objects, where $k_1+\ldots+k_r = n$ is

$$\frac{n!}{k_1!\cdots k_r!}$$

CONDITIONAL PROBABILITY

Conditional probability

Independence

Total probability

Bayes rule

Unconditional vs Conditional

- Sample space – ALL possible outcomes

- If we know that an outcome comes from a certain part of the sample space, we can discuss probability of an event **conditioning** on that part of the sample space

- Without restriction to any part of the sample space, then the probability would be **unconditional**

Example 2.6

TABLE 2.1 Sample space containing 1000 aluminum rods

	Diameter		
Length	Too Thin	OK	Too Thick
Too Short	10	3	5
OK	38	900	4
Too Long	2	25	13

Unconditional Probability

P(Diameter OK)

= 928/1000

= 0.928

TABLE 2.2 Reduced sample space containing 942 aluminum rods that meet the length specification

	Diameter		
Length	Too Thin	OK	Too Thick
Too Short	—	—	—
OK	38	900	4
Too Long	—	—	—

Conditional Probability

P(Diameter OK|Length OK)

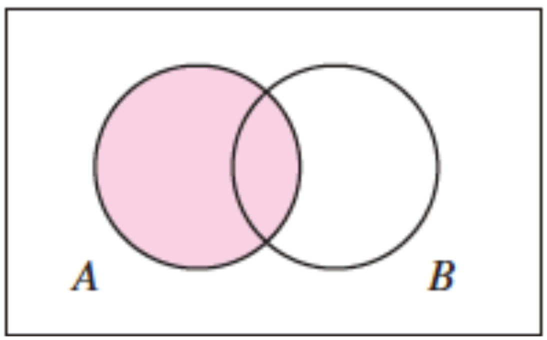
= 900/942

= 0.955

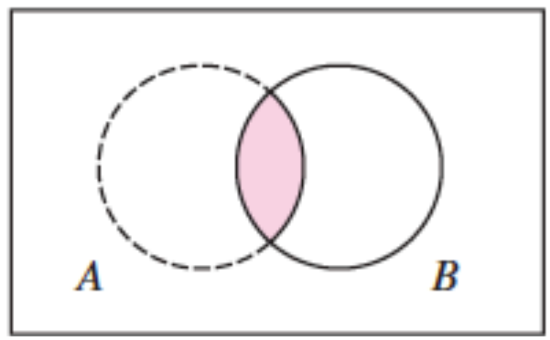
Conditional Probability Definition

Let A and B be events with $P(B) \neq 0$. The conditional probability of A given B is

$$P(A|B) = P(A \cap B) / P(B)$$



Unconditional probability of A. Ignores any information about B



Conditional probability of A **given** B occurred. Only consider the part of A that overlaps with B

Example

- Suppose that there is a 60% chance of snow (S), a 20% chance of high winds (W), and a 15% chance of snow *and* high winds together.
- What is the probability of high winds **given** that it is snowing?

  We want to find $P(W|S) = P(W \cap S) / P(S)$

  We know $P(W \cap S) = 0.15$, and

  $P(S) = 0.60$

  So $P(W|S) = 0.15 / 0.60 = 0.25$

- What is the probability of snow **given** that there are high winds?

Independent Events

- Sometimes the knowledge that event B has occurred does NOT change the probability that event A occurs, i.e.

Unconditional Probability	=	Conditional Probability

- In this case, events A and B are **independent**

Example 2.20

TABLE 2.1 Sample space containing 1000 aluminum rods

	Diameter		
Length	Too Thin	OK	Too Thick
Too Short	10	3	5
OK	38	900	4
Too Long	2	25	13

Compare P(Too long) and P(Too long| Too thin)

P(Too long) = (2+25+13)/1000 = 40/1000 = 0.04

P(Too long | Too thin) = P(Too long ∩ Too thin)/P(Too thin)
= (2/1000)/(50/1000)
= 0.04

Independence

- Two events A and B are B **independent** if

  $P(A) \neq 0$, $P(B) \neq 0$, and

  $P(B|A) = P(B)$ or $P(A|B) = P(A)$

- Generalize to n events: Events $A_1,\ldots,A_n$ are independent if the probability of each remains the same no matter which of the others occur

The Multiplication Rule

- If events A and B both have positive probability, then we can rearrange the definition of conditional probability

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

- If A and B are independent, then

$$P(A \cap B) = P(A)P(B)$$

- Generalize to multiple events: If Events $A_1,\ldots,A_n$ are independent, then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2)\ldots P(A_n)$$

Example

A certain type of light bulb has a probability of being defective of 0.02. In a random sample of 10 bulbs, what is the probability that all are functioning? Assume they are independent.

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Let L_i, i = 1,…,10 be the event that bulb i is functioning

P(All 10 function) $= P(L_1 \cap L_2 \cap \ldots \cap L_{10})$

$= P(L_1)\, P(L_2) \ldots P(L_{10})$

$= (1-0.02)^{10}$

$= (0.98)^{10}$

$= 0.817$

Law of Total Probability

- If $A_1,\ldots,A_n$ are mutually exclusive and exhaustive events, then for any event B

  $P(B) = P(B \cap A_1) + \ldots + P(B \cap A_n)$
- A set of events is **exhaustive** if their union covers the entire sample space

  example: A and A^c are mutually exclusive and exhaustive

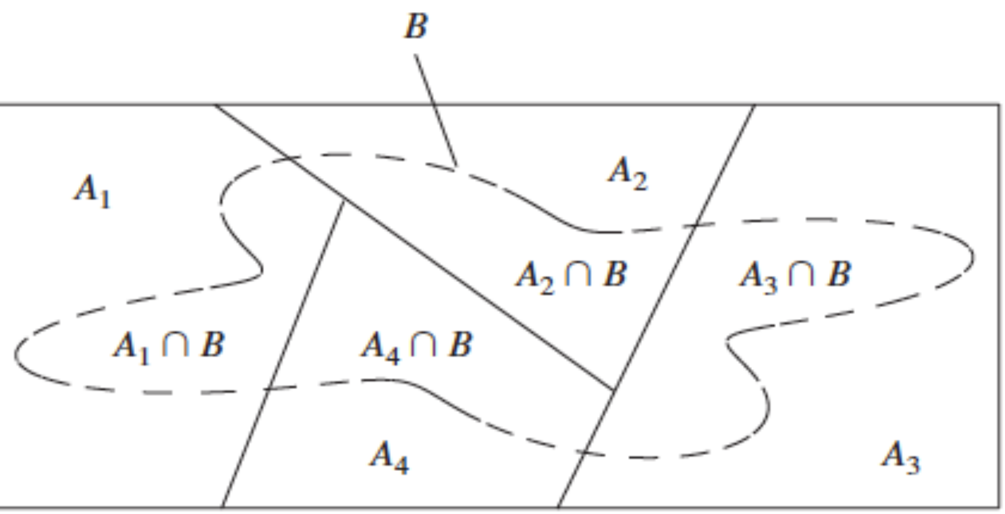


FIGURE 2.6 The mutually exclusive and exhaustive events A_1, A_2, A_3, A_4 divide the event B into mutually exclusive subsets.

Example

Experiment: throw 1 dice and toss 1 coin

S = {(1,T), (2,T), (3,T), (4,T), (5,T), (6,T),
(1,H), (2,H), (3,H), (4,H), (5,H), (6,H)}

The following events are **mutually exclusive** and **exhaustive**:

A_1 = Odd number and Tails = {(1,T), (3,T), (5,T)}
A_2 = Number > 2 and Heads = {(3,H), (4,H), (5,H), (6,H)}
A_3 = Roll a 2 = {(2,T), (2,H)}
A_4 = Roll a 1 and Heads = {(1,H)}
A_5 = Even number > 3 and Tails = {(4,T), (6,T)}

Then for an event B in the same sample space,

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) + P(B \cap A_4) + P(B \cap A_5)$$

Bayes' Rule

- Let $A_1,\ldots,A_n$ be mutually exclusive and exhaustive events with $P(A_i) \neq 0$ for each A_i. Then for any event B with $P(B) \neq 0$

$$P(A_k \mid B) = \frac{P(B \mid A_k)P(A_k)}{\sum_{i=1}^{n} P(B \mid A_i)P(A_i)}$$

- Special case for two events ($A_1=A$ and $A_2=A^c$)

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)}$$

Snow/Wind Example

- Recall our previous setup: there is a 60% chance of snow (S), and we found that $P(W|S)=0.25$. Suppose $P(W|S^c)=0.125$.
- Using Bayes rule, find the probability of snow **given** that there are high winds

$$P(S|W) = P(W|S)P(S) / [P(W|S)P(S) + P(W|S^c)P(S^c)]$$

$$= (0.25)(0.6) / [(0.25)(0.6) + 0.125(1 - 0.6)]$$

$$= 0.75$$

Example 2.26

- The proportion of people in a given community with a certain disease is 0.005 (0.5%).
- There is a test available to diagnose the disease
- If a person has the disease, the probability that the test will produce a (true) positive signal is 0.99
- If a person does not have the disease, the probability that the test will produce a (false) positive signal is 0.01

Example 2.26 Continued

If a person tests positive, what is the probability that the person actually has the disease?

Example 2.26 Continued

If a person tests positive, what is the probability that the person actually has the disease?

Given $P(D)=0.005$, $P(+|D)=0.99$, and $P(+|D^c)=0.01$

Using Bayes rule,

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|D^c)P(D^c)}$$

$$= (0.99)(0.005) / [(0.99)(0.005) + (0.01)(0.995)]$$

$$= 0.332$$

Example 2.26 Continued

If a person tests negative, what is the probability that the person actually has the disease?

Example 2.26 Continued

If a person tests negative, what is the probability that the person actually has the disease?

Given $P(D)=0.005$, $P(+|D)=0.99$, and $P(+|D^c)=0.01$

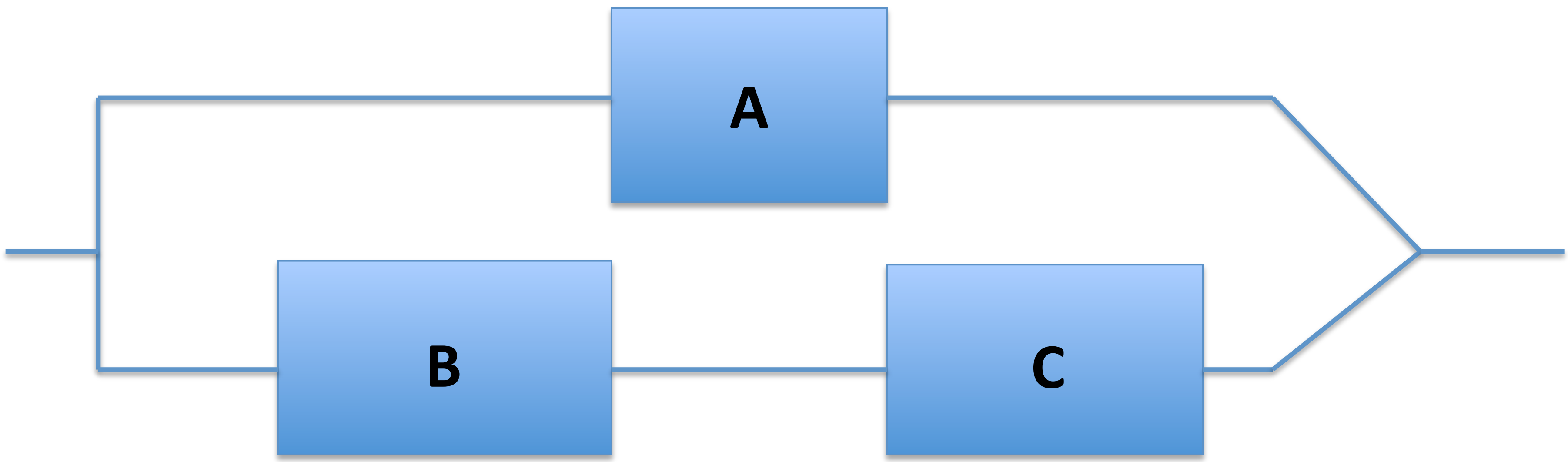
Using Bayes rule,

$$P(D|-) = \frac{P(-|D)P(D)}{P(-|D)P(D) + P(-|D^c)P(D^c)}$$

$$= (0.01)(0.005) / [(0.01)(0.005)+ (0.99)(0.995)]$$

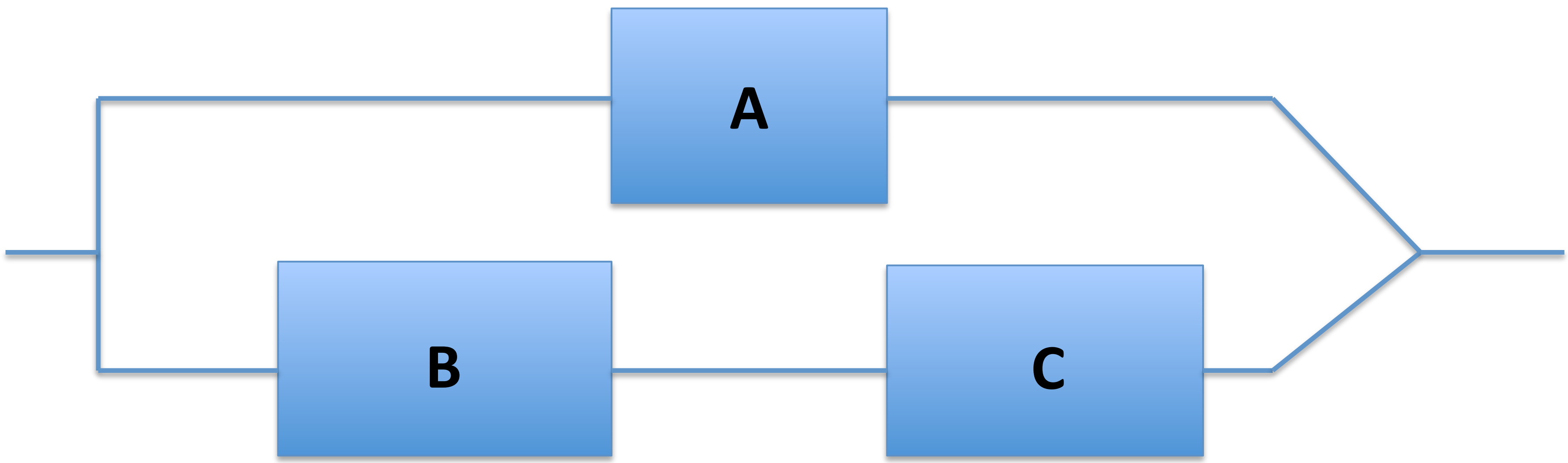
$$= 0.00005$$

Example – Reliability Analysis



- A system contains independent components A, B, and C, connected as above
- The system functions if either path functions (or both)
- The probabilities of failure are A: 0.02, B: 0.05, C: 0.01
- Find the probability that the system functions

Example – Reliability Analysis



P(System Functions) = P({A functions} or {B and C function})

$= P(A \cup (B \cap C))$ use $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B \cap C) - P(A \cap B \cap C)$ use independence

$= 0.98 + (0.95)(0.99) - (0.98)(0.95)(0.99)$

≈ 0.9988

Next

- Random variables (2.4)
  - Continuous and Discrete
  - Probability Distributions
  - Mean and Variance of a RV